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ON THE HERTZ PROBLEM FOR LINEARLY AND NONLINEARLY ELASTIC BODIES OF FINITE DIMENSIONS

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A new formulation of the contact problem is proposed and given a foundation, which would permit use of modern optimization theory methods (nonlinear programing) for its solution. An expanded exposition of the results announced in the author's note [1] is given in the paper.

In this paper the Hertz problem is understood to be the problem of determining the pressure on a part of the surface of a finite deformable body contiguous to an absolutely stiff stamp, of determining the shape and size of the contact area, and also of finding the state of stress and strain within the body. The proposed formulation differs from the majority of the problem formulations ordinarily used when either (a) the contact zone is fixed in advance and part of the components of the displacement vector of points of the surface in contact and part of the stress vector components are given in this zone, or (b) when a function connecting the normal pressure with the corresponding displacement is given [2—4]. The difficulty in the problem under consideration is that it is nonlinear even for linearly-elastic media, and the nonlinearity is determined by the boundary conditions on the part of the surface which can be in contact; these boundary conditions have the form of inequalities.

Signorini [5] first considered a similar type of elasticity theory problems; there is a survey of the latest achievements in such problems in the book [6].

The Lions-Stampacchia method of variational inequalities [7] is a comparatively simple but powerful instrument for the investigation of the problems mentioned. This method, which permits posing and studying both problems of the Signorini type as well as all possible optimization problems by a single method, thereby establishes an analogy between these two classes of problems and therefore affords the possibility of using effective methods of solving optimization problems to solve problems of Signorini type.

This paper is limited to an investigation of the contact problem for smooth stamps, i.e. stamps whose surface has a continuously rotating tangent plane (in the zone where contact is possible); cases of a linearly elastic and a physically nonlinear elastic medium are examined.

1. Formulation of the problem. Let an elastic body occupy the domain Ω with boundary S in a three-dimensional space R_3. We assume that the boundary S consists of three pieces: $S = S_u + S_\sigma + S_c$. The displacement will be considered known on the part S_u (for simplicity we assume them to be zero, and $S_u \neq \varnothing$); on the part S_σ the stresses are

$$\sigma_{ij}\nu_j \mid_{S_\sigma} = P_i \tag{1.1}$$

where σ_{ij} are the stress tensor components, and ν_i are vector components of the unit external normal to S_σ.

To describe the conditions on S_c, we assume that the boundary of the absolutely stiff stamp which can adjoin the domain Ω on the part of the surface S_c is given by the equation

$$\Psi(x) = 0 \tag{1.2}$$

where $\Psi(x) < 0$ within, and $\Psi(x) > 0$ outside the stamp.

Denoting the displacement of points of the body Ω by $u(x)$, and the stress vector on the surface S_c by σ, we use the representation $\sigma = \sigma_T + \nu\sigma_N$. Evidently, $\sigma_N = \sigma_{ij}\nu_i\nu_j$, $(\sigma_T)_i = \sigma_{ij}\nu_j - \sigma_N\nu_i$. The conditions on S_c are specified in the following manner: if points of the surface S_c belong to the surface (1.2) after deformation of the body Ω, i.e.,

$$\Psi(x + u(x)) = 0, \; x \in S_c \tag{1.3}$$

then the tangential component σ_T of the vector σ equals zero at this point, and the normal component σ_N is not positive; in the opposite case

$$\Psi(x + u(x)) > 0, \quad x \in S_c \tag{1.4}$$

the vector σ is zero at this point.

We emphasize that it is not known in advance whether the point $x + u(x)$ belongs to the surface (1.2) and this is the basic difficulty of the problem.

Later we shall assume that the quantity $|u(x)|$ is sufficiently small, $|\operatorname{grad}\Psi(x)| > 0$, $\forall x \in S_c$, and that the second derivatives of $\Psi(x)$ are bounded, so that linearization of the conditions on S_c with respect to u can be carried out

$$\begin{aligned} &\Psi(x) + u(x)\operatorname{grad}\Psi(x) = 0 \Rightarrow \sigma_T = 0, \quad \sigma_N \leqslant 0 \\ &\Psi(x) + u(x)\operatorname{grad}\Psi(x) > 0 \Rightarrow \sigma_{ij}\nu_j = 0 \\ &x \in S_c \end{aligned} \tag{1.5}$$

Let us note that always

$$\Psi(x) + u(x)\operatorname{grad}\Psi(x) \geqslant 0, \quad \forall x \in S_c \tag{1.6}$$

and this condition is a supplement to the general definition of kinematically achievable displacement fields when going over to the variational formulation of the problem, which is indeed the subject of the subsequent analysis.

Note 1.1. The Signorini problem [5] is obtained as a particular case of that under consideration for $\Psi(x) = 0$, $\operatorname{grad}\Psi(x) = -\nu\,|\operatorname{grad}\Psi(x)|$, i.e. when the initial contact zone is fixed prior to application of the external effects, the contact zone can only be diminished after deformation.

2. Reduction of the problem to a variational problem. We examine the geometrically linear problem assuming the following relationship between the stresses and strains [8]:

$$\begin{aligned} &\sigma_{ij} = \lambda\theta\delta_{ij} + 2\mu\varepsilon_{ij} - 2\mu\omega(e_u) \\ &\omega(e_u) = 1 - \frac{\Phi(e_u)}{3\mu e_u}, \quad \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \theta = \varepsilon_{ii} \end{aligned} \tag{2.1}$$

Here λ, μ are Lamé parameters, e_u is the strain intensity, the repeated subscripts denote summation within the limits 1 to 3, the comma denotes differentiation with respect to the appropriate coordinate. The case of a physically nonlinear isotropic medium follows from (2. 1) for $\omega = 0$, the generalization for anisotropic media is obvious

$$\sigma_{ij} = a_{ijkh}\, \varepsilon_{kh}$$

where a_{ijkh} is the tensor of the elastic moduli which possesses the usual symmetry and positive-definiteness properties.

Let us introduce the S. L. Sobolev space of V functions possessing the generalized first derivatives, which are square summable and vanish on S_u and the subset K in this space

$$\|v\| = \Big(\int_\Omega v_i v_i\, d\Omega + \int_\Omega v_{i,\,k} v_{i,\,k}\, d\Omega\Big)^{1/2}$$

$$K = \{v \mid v \in V;\ \ \Psi(x) + \operatorname{grad} \Psi(x)\, v(x) \geqslant 0,\ \ \forall x \in S_c\}$$

The subset K is closed (the result of the Lions theorem on traces) and convex in V (the convexity is verified directly).

The initial problem is to determine the vector-function $u = u(x)$ which satisfies the equilibriun differential equation

$$(\lambda + \mu) \operatorname{grad} \operatorname{div} u + \mu \Delta u - 2\mu \operatorname{div} [\omega(e_u)\, e_D] + \rho F = 0 \qquad (2.2)$$

(Δ is the Laplace operator, e_D is the strain tensor deviator, and ρF is the given volume force density), the boundary conditions (1. 1), (1. 5) and $u = 0$ on S_u. A set of points $x \in S_c$ for which the constraint (1. 6) is satisfied with the strict equality sign will be determined during the solution of this problem. This set is the required contact zone, and the reactive pressure in this zone is the required contact pressure.

Let us introduce the bilinear $a(u, v)$ (in the version under consideration $a_{ijkh} = \lambda\delta_{ij}\delta_{kh} + \mu(\delta_{ik}\delta_{jh} + \delta_{ih}\,\delta_{jk})$, and δ_{ij} are the Kronecker symbols), the linear $L(v)$, and the nonlinear $j(v)$ functionals in V

$$a(u, v) = \int_\Omega a_{ijkh} \varepsilon_{kh}(v)\, \varepsilon_{ij}(u)\, d\Omega$$

$$L(v) = \int_\Omega \rho F v\, d\Omega + \int_{S_\sigma} P v\, dS$$

$$j(v) = 3\mu \int_\Omega \Big(\int_0^{e_u(v)} \omega(s)\, s\, ds \Big) d\Omega$$

Theorem 2. 1. The solution of the formulated initial problem is the solution of the variational inequality

$$a(u, v - u) - j'(u, v - u) \geqslant L(v - u), \qquad \forall v \in K \ \ u \in K \qquad (2.3)$$

where $j'(u, v - u)$ denotes the functional derivative (Gato) at the point u with respect to the direction $v - u$ (Gato differentiability is evident).

Conversely, the solution of the variational inequality (2. 3) satisfies Eq. (2. 2) and the conditions (1. 1), (1. 5) (if the derivatives in (2. 2), (1. 1) and (1. 5) are meaningful).

Proof. Let u be the solution of the problem (2. 2), (1. 1), (1. 5). We multiply (2. 2) scalarly by $v - u$, where v is an arbitrary element of K, we integrate the expression obtained over Ω and we apply the Gauss-Ostrogradskii formula. We find

$$a(u, v - u) = L(v - u) + \int_{S_c} \sigma_{ij}(u)(v_i - u_i)\, \nu_j\, dS + j'(u, v - u)$$

It is seen that $\sigma_{ij}(u)(v_i - u_i)\nu_j = \sigma_T(v_T - u_T) + \sigma_N(v_N - u_N)$; according to condition (1.5) $\sigma_T = 0$ always, therefore

$$\sigma_{ij}(u)(v_i - u_i)\nu_j = \sigma_N(u)(v_N - u_N) \tag{2.4}$$

Upon compliance with the second of conditions (1.5), the right side of (2.4) equals zero; if the first of conditions (1.5) holds, then

$$\Psi + u \operatorname{grad} \Psi = 0 \tag{2.5}$$

By definition $v \in K$, therefore

$$\Psi + v \operatorname{grad} \Psi \geqslant 0 \tag{2.6}$$

Subtracting the equality (2.5) from the inequality (2.6), we obtain

$$(v - u) \operatorname{grad} \Psi \geqslant 0, \qquad \forall v \in K \tag{2.7}$$

Since contact occurs at the point under consideration, then

$$\operatorname{grad} \Psi = -\nu\gamma(x), \quad \gamma(x) \equiv |\operatorname{grad} \Psi(x)| > 0 \tag{2.8}$$

Substituting (2.8) into (2.7), we arrive at the conclusion that $u_N - v_N \geqslant 0$ at the point x under consideration, for all $v \in K$ and we have, by virtue of $\sigma_N \leqslant 0$

$$\int_{S_c} \sigma_{ij}(u)(v_i - u_i)\nu_j\, dS = \int_{S_c} \sigma_N(v_N - u_N)\, dS \geqslant 0, \quad \forall v \in K$$

This last inequality permits the conclusion that the element u satisfies the variational inequality (2.3).

Conversely, let u be the solution of the problem (2.3) and let $\varphi = \varphi(x)$ be a regular function with a compact support in Ω [9, 10]. It is evident that $u \pm \varphi \in K$; substituting the sum $u + \varphi$ first in (2.3), and then the difference $u - \varphi$, we arrive at the equality

$$a(u, \varphi) - j'(u, \varphi) = \int_\Omega \rho F \varphi\, d\Omega$$

from which it follows that the differential equation (2.2) is satisfied in the sense of the theory of distributions.

To give a foundation to the validity of the reasoning presented above, it is sufficient to assume that $\rho F_i \in L_2(\Omega)$, $\sigma_{ij,j} \in L_2(\Omega)$; this latter assumption is satisfied if $u \in H^2(\Omega) \cap V$ and $\omega(e_u)$ has a bounded first derivative. To prove that the solution of the variational inequality (2.3) satisfies the boundary conditions on S_σ and S_c, we assume in addition that $P \in L_2(S_\sigma)$ and that $S_c \subset S_\sigma$. We multiply (2.2) scalarly (in the $L_2(\Omega)$ sense) by an arbitrary element $v \in K$ and we find by applying the Gauss-Ostrogradskii formula (the assumptions about smoothness which were formulated above are used in this step)

$$a(u, v) - j'(u, v) = \int_\Omega \rho F v\, d\Omega + \int_{S_\sigma \cup S_c} \sigma_{ij}(u)\nu_j v_i\, dS \tag{2.9}$$

Setting $v = u$ here, subtracting from the expression (2.9) obtained and using the variational inequality (2.3), we find

$$\int_{S_\sigma} [\sigma_{ij}(u)\nu_j - P_i](v_i - u_i)\, dS + \int_{S_c} \sigma_{ij}(u)\nu_j(v_i - u_i)\, dS \geqslant 0, \quad \forall v \in K \tag{2.10}$$

Subjecting the choice of v in (2.10) to the additional constraint $v = u$ on S_c, we obtain in place of (2.10)

$$\int_{S_\sigma} [\sigma_{ij}(u)\nu_j - P_i](v_i - u_i)\,dS \geqslant 0, \quad \forall v \in K, \quad v = u \text{ на } S_c \quad (2.11)$$

Let φ denote the trace of the function $v - u$ on S_σ, the set $\{\varphi\}$ is the linear subspace in $H^{1/2}(S_\sigma)$ [11]; evidently $\varphi = 0$ on ∂S_σ is the boundary of the manifold S_σ.

We first consider the inequality (2.11) in the case when φ runs through the set $D(S_\sigma)$ of infinitely differentiable functions with the compact support in S_σ. It is seen that

$$\int_{S_\sigma} [\sigma_{ij}(u)\nu_j - P_i]\,\varphi_i\,dS = 0, \quad \forall \varphi \in D(S_\sigma) \quad (2.12)$$

It is known [11] that $D(S_\sigma)$ is compact in $L_2(S_\sigma)$, therefore, there results from (2.12) that

$$\sigma_{ij}(u(x))\,\nu_j = P_i(x), \quad x \in S_\sigma \quad (2.13)$$

as an element of the space $L_2(S_\sigma)$. It follows from the condition (2.13) obtained, the inequality (2.10) and the continuity of the mapping $H^1(\Omega)$ in $L_2(S)$ [11] that

$$\int_{S_c} \sigma_{ij}(u)\nu_j(v_i - u_i)\,dS \geqslant 0, \quad \forall v \in K \quad (2.14)$$

Let us set $v = u + \varepsilon\varphi$, where ε, is an arbitraty positive number; by using the expansion $\varphi = \varphi_N\,\nu + \varphi_T$, we rewrite the inequality (2.14) in the form

$$\int_{S_c} \sigma_N(u)\,\varphi_N\,dS + \int_{S_c} \sigma_T(u)\,\varphi_T\,dS \geqslant 0 \quad (2.15)$$

Let there be contact at the point $x \in S_c$ under consideration; taking account of (2.8), we find that at this point $\varphi_N(x) \leqslant 0$, while $\varphi_T(x)$ is arbitrary. If there is no contact at the point $x \in S_c$, then

$$\varepsilon\varphi(x)\,\mathrm{grad}\,\Psi(x) \geqslant -[\Psi(x) + u(x)\,\mathrm{grad}\,\Psi(x)] \quad (2.16)$$

where the right side of this inequality is strictly negative; therefore, for sufficiently small ε the function $\varphi(x)$ (and $\varphi_T(x)$) is arbitrary.

Now, we note that the set

$$W = \{\varphi \mid \varphi = \varphi(x),\ x \in S_c,\ \varphi_N(x) = 0\}$$

is a linear subspace in $L_2(S_c)$; considering the inequality (2.15) in $W \cap D(S_c)$, we conclude that

$$\int_{S_c} \sigma_T(u)\,\varphi_T\,dS = 0, \quad \forall \varphi \in W \cap D(S_c) \quad (2.17)$$

Taking account of the density of the embedding of $D(S_c)$ in $L_2(S_c)$, we conclude that

$$\sigma_T(u(x)) = 0, \ \forall x \in S_c \quad (2.18)$$

as an element of $L_2(S_c)$. It follows from (2.18) and the continuity of the mapping $H^1(\Omega)$ in $L_2(S)$ that the inequality (2.15) has the form

$$\int_{S_c} \sigma_N(u)(v_N - u_N)\,dS \geqslant 0, \quad \forall v \in K \quad (2.19)$$

Let S_c^v denote the contact zone, i.e. the set of those points $x \in S_c$ for which (1.6) is converted into the equality; again let φ_N denote the trace of the difference $v_N - u_N$ on S_c and let us subject the selection of φ_N to the additional constraint $\varphi_N = 0$ on S_c^v.

We see by the method used twice above, that

$$\sigma_N(x) = 0, \quad \forall x \in S_c - S_c^v$$

as an element of the space L_2. We have hence and from the inequality (2.19)

$$\int_{S_c^v} \sigma_N(u)\,\varphi_N\,dS \geqslant 0, \quad \forall \varphi_N \in L_2(S_c^v), \quad \varphi_N \leqslant 0$$

It hence follows that

$$\sigma_N(u(x)) \leqslant 0, \quad \forall x \in S_c^v$$

which completes the proof of Theorem (2.1).

Note 2.1. The transformations performed in the proof of Theorem 2.1 remain valid even for $P_i \in H^{-1/2}(S_\sigma)$ [11].

Note 2.2. The properties of continuity and the density of embeddings of one space into the other were used in proving the second part of Theorem 2.1. These properties are true only under definite constraints on the shape and smoothness of the boundary of the domain of definition; the establishment of these constraints is an open problem [12].

Theorem 2.2. The solution of the variational inequality (2.3) is equivalent to the problem of minimizing the functional

$$J(v) = {}^1/_2 a(v, v) - L(v) - j(v), \quad v \in K \tag{2.20}$$

On the basis of the known results in [13], the proof of this theorem reduces to verification of the convexity and differentiability of $J(v)$; this verification will be performed in Sect. 3 in the investigation of the existence and uniqueness.

3. Existence and uniqueness of the solution.

Theorem 3.1. If the function $\omega(s)$ is continuous and

$$0 \leqslant \omega < d(s\,\omega(s))\,/\,ds < 1, \quad d\,\omega(s)\,/\,ds \geqslant 0 \tag{3.1}$$

$$\rho F_i \in L_2(\Omega), \quad P_i \in H^{-1/2}(S_\sigma) \tag{3.2}$$

then the problem of minimizing the functional (2.20) formulated in Sect. 2 has a solution, and moreover, only one.

The proof is based on the theorem [14, 15] about the existence and uniqueness of the solution of the problem of convex nonlinear programing and reduces to verification of the strict convexity and coercitivity of the functional (2.20); the coercitivity means that

$$\lim J(v) = +\infty, \quad \|v\| \to +\infty \tag{3.3}$$

We prove the coercitivity without using condition (3.1) by assuming concavity and monotonic growth of the function $\Phi(e_u)$.

Lemma 3.1. If $\Phi(e_u)$ is a strictly concave, monotonically increasing function, where $\Phi(e_u) \leqslant 3\,\mu e_u$, $\forall e_u$, then we have the estimate

$$\int_\Omega \Big[\int_0^{e_u(v)} \omega(x)\,x\,dx\Big] d\Omega \leqslant c\,\|v\|^{2-\varepsilon}, \quad 0 < \varepsilon < 1 \tag{3.4}$$

where c and ε are some positive constants.

Proof. By the method of contradiction it can be verified that there exists a $0 < \varepsilon < 1$ such that

$$\omega(x)\, x \leqslant x^{1-\varepsilon}, \quad \forall x \in [a, b], \quad \forall [a, b] \tag{3.5}$$

It follows from (3.5)

$$\int\limits_0^{e_u(v)} \omega(x)\, x\, dx \leqslant c_1 [e_u(v)]^{2-\varepsilon}, \quad c_1 = \frac{1}{1-\varepsilon} \tag{3.6}$$

Now, we prove that

$$\int\limits_\Omega e_u(v)\, d\Omega \leqslant c_2 \| v \|_V, \quad c_2 = \text{const} > 0 \tag{3.7}$$

In fact, by definition

$$\| v \|_V^2 = \int\limits_\Omega v_i v_i\, d\Omega + \int\limits_\Omega v_{i,j} v_{i,j}\, d\Omega \geqslant \int\limits_\Omega v_i v_i\, d\Omega + \int\limits_\Omega \varepsilon_{ij}(v)\, \varepsilon_{ij}(v)\, d\Omega = \tag{3.8}$$

$$\int\limits_\Omega v_i v_i\, d\Omega + \int\limits_\Omega e_{ij}(v)\, e_{ij}(v)\, d\Omega + \int\limits_\Omega \frac{1}{3} \theta^2(v)\, d\Omega \geqslant \int\limits_\Omega \frac{3}{2} e_u^2\, d\Omega$$

By the Cauchy-Buniakowski inequality

$$\int\limits_\Omega e_u\, d\Omega \leqslant \Big(\int\limits_\Omega e_u^2\, d\Omega\Big)^{1/2} \Big(\int\limits_\Omega d\Omega\Big)^{1/2} = (\text{mes }\Omega)^{1/2} \Big(\int\limits_\Omega e_u^2\, d\Omega\Big)^{1/2}$$

Hence, and from the inequality (3.8) there follows (3.7) where $c_2 = \sqrt{2/3}\, (\text{mes }\Omega)^{1/2}$.

To complete the proof of the lemma, we use the Hölder inequality

$$\int\limits_\Omega \varphi_1 \varphi_2\, d\Omega \leqslant \Big(\int\limits_\Omega \varphi_1^p\, d\Omega\Big)^{1/p} \Big(\int\limits_\Omega \varphi_2^q\, d\Omega\Big)^{1/q}$$

$$\frac{1}{p} + \frac{1}{q} = 1, \quad p > 0, \quad q > 0$$

Setting $p = 2/(2-\varepsilon)$, $q = 2/\varepsilon$, $\varphi_1 = e_u^{2-\varepsilon}$, $\varphi_2 = 1$ therein, we find

$$\int\limits_\Omega e_u^{2-\varepsilon}\, d\Omega \leqslant \Big(\int\limits_\Omega e_u\, d\Omega\Big)^{(2-\varepsilon)/2} \tag{3.9}$$

Combining the estimates (3.6), (3.7) and (3.9), we obtain (3.4).

Now, we prove the coercitivity of $J(v)$; let us first note that the elastic modulus tensor is

$$a_{ijkh}\, \varepsilon_{kh}\, \varepsilon_{ij} \geqslant \beta \varepsilon_{ij}\, \varepsilon_{ij}, \quad \forall \varepsilon_{ij} = \varepsilon_{ji}, \quad \beta = \text{const} > 0$$

from the inequality of positive definiteness, while the estimate

$$a(v, v) \geqslant \alpha^2 \| v \|^2, \quad \forall v \in V, \ \alpha \neq 0 \tag{3.10}$$

follows from the Korn inequality.

The form $L(v)$ is evidently continuous on V, therefore

$$| L(v) | \leqslant \| L \| \, \| v \|$$

Hence

$$J(v) = {}^1/_2\, a(v, v) - L(v) - j(v) \geqslant {}^1/_2\, \alpha^2 \| v \|^2 -$$

$$\| L \| \, \| v \| - c \| v \|^{2-\varepsilon} \to +\infty \quad \text{for } \| v \| \to +\infty$$

and the coercitivity of $J(v)$ is therefore established.

Note 3.1. The computations and reasoning performed do not go through if the set S_u is empty. To prove the estimate (3.10), and the existence and uniqueness together with (3.10), it must be assumed that

$$\int_\Omega \rho F v \, d\Omega + \int_{S_\sigma} P v \, dS < 0, \quad \forall v = a + [b \times x]$$

where a and b are constant vectors, $v \neq 0$, $v \in K$. This condition is known as the strong Signorini hypothesis [7, 12].

Note 3.2. The existence theorem was first proved in the ordinary elasticity-plasticity problem in [16], where a condition of the form (3.1) was also used.

Note 3.3. It is necessary to assume that $\Phi(0) = 0$ in proving Lemma 3.1; this condition is satisfied for materials without initial stresses.

We prove the convexity under the additional assumption on the existence of $j''(u, \varphi, \varphi)$. We have

$$j''(u, \varphi, \varphi) = 2\mu \int_\Omega [\omega(e_u(u))\, {}^3/_2 e_u^2(\varphi) +$$

$$({}^2/_3 \omega'(e_u(u)) / e_u(u)) (e_{ij}(u) e_{ij}(\varphi))^2] \, d\Omega$$

According to the Cauchy-Buniakowski inequality

$$[e_{ij}(u)\, e_{ij}(\varphi)]^2 \leqslant {}^9/_4\, e_u^2(u)\, e_u^2(\varphi) \tag{3.11}$$

Using (3.11) and (3.1), we see that

$$j''(u, \varphi, \varphi) < a(\varphi, \varphi)$$

Hence, by virtue of the criterion [13], strict convexity of the functional (2.12) follows, which indeed competes the proof of Theorem 3.1.

Note 3.4. Theorem 3.1 is valid and its proof is simplified when $\Phi(e_u)$ is a linear function which grows no more rapidly than $3\mu e_u$.

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